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Appearance of vertices of infinite order in a model of random trees

Thordur Jonsson and Sigurður Örn Stefánsson

The Science Institute, University of Iceland, Dunhaga 3, 107 Reykjavik, Iceland

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Abstract
We study an equilibrium statistical mechanical model of tree graphs which are made up of a linear subgraph (the spine) to which leaves are attached. We prove that the model has two phases, a generic phase where the spine becomes infinitely long in the thermodynamic limit and all vertices have finite order and a condensed phase where the spine is finite with probability 1 and a single vertex of infinite order appears in the thermodynamic limit. We calculate the spectral dimension of the graphs in both phases and prove the existence of a Gibbs measure. We discuss generalizations of this model and the relationship with models of nongeneric random trees.

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1. Introduction

The study of random graphs has been an active area of research in mathematics and physics for the past few decades and remains so. In particular, the study of random trees and random triangulations has found many applications in theoretical physics, see e.g. [1]. Our understanding of the equilibrium statistical mechanics of trees with local action is fairly good but not complete. By local action we mean an action which is given by a sum over the vertices and only depends on their order. It is now known that the so-called generic trees can be viewed as critical Galton–Watson processes [13] which are very well understood mathematically [3]. A corresponding picture has not been fully established for nongeneric trees which are more difficult to analyse. Much of our knowledge about such trees comes from numerical simulations and educated guesswork [4–6, 8, 9]. However, a consistent picture has emerged [20]. Typically, a vertex of infinite order appears in the thermodynamic limit but full analytic control of this phase of random trees is still missing.

In this paper, we study a simple model of random graphs which exhibits the same behaviour as random trees with a local action, namely there is a generic phase where the free energy can be calculated by a saddle point technique and a nongeneric phase where a vertex of infinite order appears in the thermodynamic limit. This model was analysed extensively some years ago in a series of papers [4–6] under the name 'balls in boxes' and 'backgammon' model.

Closely related models appear in the study of the equilibrium distribution for urn models and zero-range processes, see e.g. [15, 17] and references therein.

The graphs that underlie the model studied in this paper have been called caterpillar graphs or simply caterpillars by graph theorists [18] and we will adopt that name here. Caterpillars are defined as graphs with the property that all vertices of order higher than 1 form a linear subgraph, i.e. if all leaves are removed one ends up with a linear graph. Various applications of caterpillar graphs in physics and chemistry are described in [14]. A recent model of random trees, the alpha–gamma model [10], includes as a special case ($\alpha = 1$) a model of random caterpillar graphs.

When the caterpillar grows large, two things can happen: it either becomes very long or some of the vertices will have a large number of leaves. *A priori,* these two phenomena could coexist but we will see that this is not the case in the model we consider. Our main motivation is to study the appearance of a vertex of infinite order in a rigorous fashion.

This paper is organized as follows. In the next section we define the model, establish our notation and derive some simple properties. In section 3, we study the generic phase and prove that generic caterpillars are infinitely long in the thermodynamic limit with all vertices of finite order. We calculate the order distribution explicitly. The Hausdorff and spectral dimensions of generic caterpillars are both shown to be equal to 1. In section 4, which is the core of this paper, we study nongeneric caterpillars and begin by establishing an asymptotic formula for the canonical partition function. We then prove that there arises exactly one vertex of infinite order in the thermodynamic limit. We find the probability distribution of the distance from the root of the random caterpillar (taken to be one of the endpoints of the spine) to the infinite order vertex as well as the probability distribution for the orders of the other vertices.

The nongeneric caterpillar graphs have infinite Hausdorff and spectral dimensions since there is a vertex of infinite order at a finite distance from the root with probability 1. However, we will show that the spectral dimension defined in terms of the ensemble average of the return probability of random walker is finite and varies continuously with the parameters of the model.

In section 5, we comment on generalizations of this model and discuss nongeneric trees and how they are related to the caterpillar model. In an appendix, we establish the existence of a probability measure on the set of infinite caterpillar graphs where vertices may have infinite order.

2. The model

A finite caterpillar is a finite graph which consists of a linear graph, which we call the spine, to which leaves (i.e. individual links) are attached. We mark the end vertices of the linear graph by r_1 and r_2 and call r_1 the root of the caterpillar. Both these vertices have order 1 by definition. Furthermore, we will view the caterpillars as planar graphs so we distinguish between left leaves and right leaves, see figure 1. The assumption of planarity is not essential. We denote the set of all caterpillars with N edges by B_N. For a caterpillar $\tau \in B_N$, denote the graph distance between r_1 and r_2 by $\ell(\tau)$ and call it the length of the caterpillar. For a caterpillar of length ℓ, we denote the vertices on the spine between r_1 and r_2 by $s_1, \ldots, s_{\ell-1}$.

Let w_n, $n = 1, 2, \ldots$, be a sequence of nonnegative numbers which will be called weight factors. The weight of a caterpillar $\tau \in B_N$ is defined as

$$w(\tau) = \prod_{i \in \tau \setminus \{r_1, r_2\}} w_{\sigma(i)}, \tag{1}$$

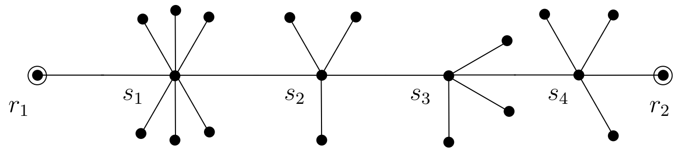

Figure 1. An example of a finite caterpillar graph.

where $\sigma(i)$ denotes the order of the vertex i and by abuse of notation we let τ also denote the set of vertices in τ. We define the finite volume partition function by

$$Z_N = \sum_{\tau \in B_N} w(\tau) \tag{2}$$

and a probability distribution on B_N by

$$\nu_N(\tau) = \frac{w(\tau)}{Z_N}. \tag{3}$$

The weight factors w_n, or alternatively the measures ν_N, define what we call a caterpillar ensemble.

Since the probability of a given caterpillar only depends on the order of its vertices, an equivalent way of defining this ensemble is the following. If $\tau \in B_N$ consider the finite sequence $c(\tau) = (\sigma(s_1), \sigma(s_2), \ldots, \sigma(s_{\ell-1}))$ and assign to it the probability

$$\tilde{\nu}_N(c(\tau)) = \nu_N(\tau) \prod_{i=1}^{\ell(\tau)-1} (\sigma(s_i) - 1). \tag{4}$$

The product factor in (4) accounts for the number of different caterpillars which correspond to the same sequence $c(\tau)$. Define the set $\tilde{B}_N = \{c(\tau) | \tau \in B_N\}$. It is clear that (B_N, ν_N) is equivalent to $(\tilde{B}_N, \tilde{\nu}_N)$ in the sense that $\nu_N(\tau)$ only depends on $c(\tau)$. This allows us to extend the notion of finite caterpillars to infinite ones:

$$\tilde{B} = \left\{ (b_i)_{i=1}^{k-2} \big| k, b_i \in \{2, 3, \ldots\} \cup \{\infty\}, 1 \leqslant i \leqslant k-2 \right\}, \tag{5}$$

where $k = 2$ corresponds to the unique caterpillar of length $\ell = 1$. Note that an element in $\tilde{B}$ which has infinite terms and/or infinite length has no counterpart in B_N for any N.

Define the finite volume partition function with fixed distance ℓ between r_1 and r_2 as

$$Z_{N,\ell} = \sum_{\tau \in B_N, \ell(\tau) = \ell} w(\tau). \tag{6}$$

It is useful to work with the generating functions

$$Z(\zeta) = \sum_{N=1}^{\infty} Z_N \zeta^N \tag{7}$$

and

$$g(z) = \sum_{n=0}^{\infty} w_{n+1} z^n \tag{8}$$

with radii of convergence ζ_0 and ρ, respectively, both of which we assume to be nonzero. Define also

$$\hat{Z}_\ell(\zeta) = \sum_{N=1}^{\infty} Z_{N,\ell} \zeta^N. \tag{9}$$

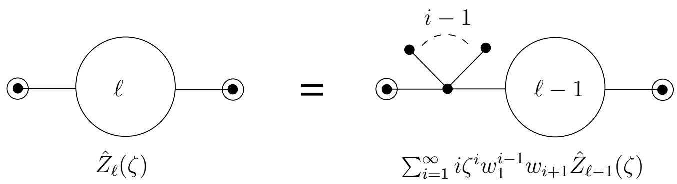

Figure 2. An illustration of the recursion (11).

Then it is clear that

$$Z(\zeta) = \sum_{\ell=1}^{\infty} \hat{Z}_\ell(\zeta). \tag{10}$$

We have the recursion relation

$$\hat{Z}_\ell(\zeta) = \zeta g'(w_1\zeta)\hat{Z}_{\ell-1}(\zeta), \tag{11}$$

for any $\ell \geqslant 1$, see figure 2.

Using the above equation and $\hat{Z}_1(\zeta) = \zeta$ gives

$$\hat{Z}_\ell(\zeta) = \zeta(\zeta g'(w_1\zeta))^{\ell-1} \tag{12}$$

and by (10),

$$Z(\zeta) = \frac{\zeta}{1 - \zeta g'(w_1\zeta)}. \tag{13}$$

From (13), we see that ζ_0 is the smallest solution of the equation

$$\zeta g'(w_1\zeta) = 1 \tag{14}$$

on the interval $(0, \rho/w_1)$ if such a solution exists. If it does not exist, then $\zeta_0 = \rho/w_1$.

If $\zeta_0 < \rho/w_1$ then g is analytic at $w_1\zeta_0$ and we say that we have a *generic ensemble*. This has been called the 'fluid phase' by other authors [6]. If $\zeta_0 = \rho/w_1$ we have a *nongeneric ensemble*. Notice that if $\rho = \infty$ then the ensemble is always generic. For nongeneric ensembles we therefore have finite ρ. In that case we can always choose $\rho = 1$ by scaling the weights $w_n \to w_n\rho^{n-1}$. This scaling does not affect the probabilities (3).

Now consider weight factors with $\rho = 1$ and let w_1 be a free parameter. The genericity condition is then $\frac{1}{w_1}g'(1) > 1$, i.e. $w_1 < w_c$ where

$$w_c \equiv g'(1) = \sum_{n=2}^{\infty}(n-1)w_n \tag{15}$$

is a critical value for w_1. If $w_1 = w_c$, we have a nongeneric ensemble which we refer to as *critical* and if $w_1 > w_c$ we have a nongeneric ensemble which we refer to as *subcritical*. This phase has been called the 'condensed phase' in the literature [6].

3. The generic phase

Let w_n be weight factors with $w_1 \neq 0$ and $w_n \neq 0$ for some $n > 2$ which lead to a generic ensemble.

Lemma 1. *Under the stated assumptions on the weight factors, the asymptotic behaviour of Z_N is given by*

$$Z_N = \frac{1}{g'(w_1\zeta_0) + \zeta_0 w_1 g''(w_1\zeta_0)} \zeta_0^{-N}(1 + O(N^{-1})) \tag{16}$$

if the integers $n > 0$ for which $w_{n+1} \neq 0$ have no common divisors greater than 1. Otherwise, if their greatest common divisor is $d \geqslant 2$, then

$$Z_N = \frac{d}{g'(w_1\zeta_0) + \zeta_0 w_1 g''(w_1\zeta_0)} \zeta_0^{-N}(1 + O(N^{-1})), \tag{17}$$

if $N = 1$ mod d, and $Z_N = 0$ otherwise.

The proof of this lemma is standard, cf [16], where the corresponding result for generic trees is established. For generic caterpillars one can show by a straightforward application of the methods of [11] (see also the appendix) that the measures $\tilde{\nu}_N$ converge as $N \to \infty$ to a measure $\tilde{\nu}$ which is concentrated on locally finite caterpillars of infinite length and the orders of the vertices on the infinite spine are independently and identically distributed by

$$\phi(n) = \zeta_0(n-1)w_n(w_1\zeta_0)^{n-2}, \qquad n \geqslant 2. \tag{18}$$

Denote the expectation with respect to the measure $\tilde{\nu}$ by $\langle\cdot\rangle_{\tilde{\nu}}$. If V_r is the number of vertices within a distance r from the root, the Hausdorff dimension d_H is defined as

$$\langle V_r\rangle_{\tilde{\nu}} \sim r^{d_H}. \tag{19}$$

We write $f(x) \sim x^\gamma$ if for any $\epsilon > 0$ there are constants C_1 and C_2 such that $C_1 x^{\gamma+\epsilon} \leqslant f(x) \leqslant C_2 x^{\gamma-\epsilon}$. If $\langle V_r\rangle_{\tilde{\nu}}$ increases faster than any power of r then we say that d_H is infinite. We see from (18) that the expectation value (19) is

$$\langle V_r\rangle_{\tilde{\nu}} = (\zeta_0 g''(w_1\zeta_0) - 1)(r-1) + 1. \tag{20}$$

It follows that the Hausdorff dimension of generic caterpillars is 1.

Let $p_\tau(t)$ be the probability that a simple random walk which leaves the root of an infinite caterpillar τ at time 0 is back at the root at time t, i.e. after t steps. If there exists a number $d_s > 0$ such that

$$p_\tau(t) \sim t^{-d_s/2} \tag{21}$$

as $t \to \infty$, then we say that the spectral dimension of the graph is d_s. If $p_\tau(t)$ decays faster than any power of t then we say that d_s is infinite. For a discussion of the spectral dimension of some random graph ensembles, see [12, 13, 19].

The spectral dimension is most conveniently analysed by generating functions. We define

$$Q_\tau(x) = \sum_{t=0}^{\infty} p_\tau(t)(1-x)^{t/2} \tag{22}$$

and let $Q(x) = \langle Q_\tau(x)\rangle_{\tilde{\nu}}$. We define $p_\tau^{(1)}(t)$ to be the probability that a simple random walk which leaves the root at time 0 is back at the root for the first time after t steps and let $P_\tau(x)$ be the corresponding generating function defined as $Q_\tau(x)$ with $p_\tau(t)$ replaced by $p_\tau^{(1)}(t)$. Then we have the relation

$$Q_\tau(x) = \frac{1}{1 - P_\tau(x)}. \tag{23}$$

Let n be the smallest nonnegative integer for which $Q_\tau^{(n)}(x)$, the nth derivative of $Q(x)$, diverges as $x \to 0$. If

$$(-1)^n Q_\tau^{(n)}(x) \sim x^{-\alpha} \tag{24}$$

for some $\alpha \in [0, 1)$, then clearly

$$d_s = 2(1 - \alpha + n), \tag{25}$$

if d_s exists. We define the spectral dimension of the caterpillar ensemble by (25) provided $(-1)^n Q^{(n)}(x) \sim x^{-\alpha}$.

From the monotonicity lemmas in [19], we get an upper bound $x^{-1/2}$ on $Q(x)$ by throwing away all the legs of the caterpillar. To get a lower bound on $Q(x)$, we use a slight modification of lemma 7 in [13] which is the following. For a given infinitely long caterpillar τ with a first return probability generating function $P_\tau(x)$ we have, for all integers $L \geqslant 1$ and $0 < x \leqslant 1$,

$$P_\tau(x) \geqslant 1 - \frac{1}{L} - x\sum_{i=1}^{L} \sigma(s_i(\tau)). \tag{26}$$

We then get, using (23), (26) and Jensen's inequality,

$$Q(x) \geqslant \frac{1}{1 - \langle P_\tau(x)\rangle_{\tilde{\nu}}} \geqslant \frac{1}{\frac{1}{L} + \langle \sigma(s_1)\rangle_{\tilde{\nu}} L x}. \tag{27}$$

In the generic phase, we see from equation (18) that $\langle \sigma(s_1)\rangle_{\tilde{\nu}}$ is finite. Choosing $L = [x^{-1/2}]$, we find

$$Q(x) \geqslant c x^{-1/2}, \tag{28}$$

where c is a constant. It follows from (28), the upper bound on $Q(x)$ and (25) that the spectral dimension of generic caterpillars is $d_s = 1$.

4. The subcritical phase

In this section, we begin by calculating the asymptotic behaviour of the canonical partition function in the subcritical phase. We then show that there is exactly one vertex of infinite order in the thermodynamic limit. The mechanism leading to a unique vertex of infinite order is similar to the one leading to a unique spine for generic trees [11, 13]. We calculate the probability distribution for the location of the infinite order vertex as well as the probability distribution for the orders of the other vertices. Finally, we discuss the spectral dimension of subcritical caterpillars.

We take $\rho = 1$ and $w_1 > w_c$ so that we are in the subcritical phase. We study a concrete model where

$$w_i = i^{-\beta}, \qquad i \geqslant 2, \tag{29}$$

and let w_1 be a free parameter in the specified range. We comment on extensions in section 5. Figure 3 shows the phase diagram of the caterpillars. A necessary condition for being in the subcritical phase is $\beta > 2$ since otherwise $w_c = \infty$.

Lemma 2. *For the weights given in (29) and $w_1 > w_c$, we have*

$$Z_N = \frac{1}{(w_1 - w_c)^2} N^{1-\beta} w_1^N (1 + o(1)) \tag{30}$$

as $N \to \infty$.

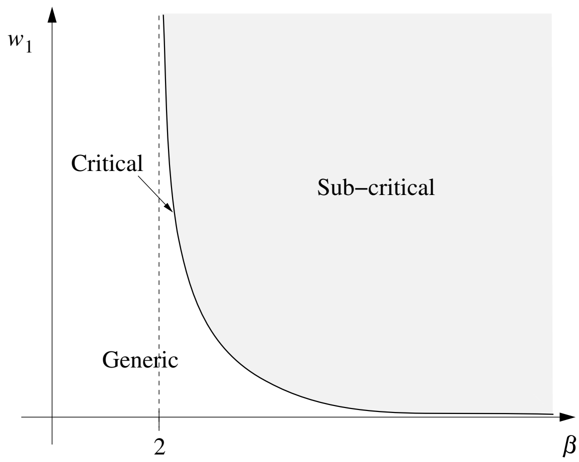

Figure 3. A diagram showing the different phases of the caterpillars.

Proof. We can write

$$Z_N = \sum_{\ell=1}^{N} Z_{N,\ell}. \tag{31}$$

Define a sequence of functions f_N on the positive integers by

$$f_N(\ell) = \begin{cases} w_1^{-N} N^{\beta-1} Z_{N,\ell} & \ell \leqslant N \\ 0 & \ell > N. \end{cases} \tag{32}$$

We claim that

$$\lim_{N\to\infty} f_N(\ell) = \frac{1}{w_c^2}(\ell-1)\left(\frac{w_c}{w_1}\right)^{\ell} \equiv f(\ell). \tag{33}$$

We accept the claim for a moment and finish the proof of the lemma.

It is clear that $f_N(\ell)$ is summable for every N. We also see that $f(\ell)$ is summable since $w_1 > w_c$. Note that for $\ell \leqslant N$,

$$\begin{aligned} f_N(\ell) &= w_1^{-\ell} N^{\beta-1} \sum_{N_1+\cdots+N_{\ell-1}=N-\ell} \prod_{i=1}^{\ell-1} \left\{(N_i+1) w_{N_i+2}\right\} \\ &\leqslant w_1^{-\ell} N^{\beta-1} (\ell-1) \sum_{\substack{N_1+\cdots+N_{\ell-1}=N-\ell \\ N_1 \geqslant \frac{N-\ell}{\ell-1}}} \frac{N_1+1}{(N_1+2)^{\beta}} \prod_{i=2}^{\ell-1} \left\{(N_i+1) w_{N_i+2}\right\} \\ &\leqslant \frac{1}{w_c^2}\left(\frac{w_c}{w_1}\right)^{\ell} \frac{N^{\beta-1}(N-1)}{\left(\frac{N-l}{\ell-1}+2\right)^{\beta}} \leqslant C(\ell-1)^{\beta}\left(\frac{w_c}{w_1}\right)^{\ell}, \end{aligned} \tag{34}$$

where C is a positive constant. The first inequality in (34) is obtained by observing that at least one of the indices N_i must be larger than $\frac{N-\ell}{\ell-1}$ and in the second one we used the definition of w_c. It follows that the sequence $\{f_N\}_1^{\infty}$ is dominated by a summable function and we can calculate the limit

$$\lim_{N\to\infty}\left(w_1^{-N} N^{\beta-1} Z_N\right) = \lim_{N\to\infty} \sum_{\ell=1}^{\infty} f_N(\ell) = \sum_{\ell=1}^{\infty} f(\ell) = \frac{1}{(w_1-w_c)^2}. \tag{35}$$

This implies the desired result.

It remains to prove the claim (33). There is at least one index i in the sum defining $f_N(\ell)$ such that $N_i \geqslant \frac{N-\ell}{\ell-1}$. If there is another index $j \neq i$ such that $N_j > A$ where $A > 1$ is a constant then we get an upper bound on that contribution to $f_N(\ell)$ of the form

$$
\begin{aligned}
w_1^{-\ell} N^{\beta-1}(\ell-1)^2 & \sum_{\substack{N_1+\cdots+N_{l-1}=N-\ell \\ N_1 \geqslant \frac{N-\ell}{\ell-1} \\ N_2>A}} \frac{N_1+1}{(N_1+2)^\beta} \prod_{i=2}^{\ell-1}\left\{(N_i+1) w_{N_i+2}\right\} \\
& \leqslant C(\ell) \frac{N^\beta}{(N+\ell-2)^\beta} \sum_{N_3, \ldots, N_{\ell-1} \geqslant 0} \prod_{i=3}^{\ell-1}\{(N_i+1) w_{N_i+2}\} \sum_{N_2>A}(N_2+1) w_{N_2+2} \\
& \leqslant D(\ell) w_c^{\ell-3} \sum_{N_2>A}(N_2+a) w_{N_2+2},
\end{aligned} \tag{36}
$$

where $C(\ell)$ and $D(\ell)$ are numbers which only depend on ℓ. The last expression goes to zero as $A \to \infty$ since $g'(1)$ is finite. The remaining contribution to $f_N(\ell)$ is

$$
\begin{aligned}
w_1^{-\ell} N^{\beta-1}(\ell-1) & \sum_{\substack{N_1+\cdots+N_{\ell-1}=N-\ell \\ N_1 \geqslant \frac{N-\ell}{\ell-1} \\ N_j \leqslant A, \quad j \neq 1}} \prod_{i=1}^{\ell-1}\left\{(N_i+1) w_{N_i+2}\right\} \\
& \underset{N\to\infty}{\longrightarrow} w_1^{-\ell}(\ell-1)\left(\sum_{n=0}^{A}(n+1) w_{n+2}\right)^{\ell-2} \\
& \underset{A\to\infty}{\longrightarrow} w_c^{-2}(\ell-1)\left(\frac{w_c}{w_1}\right)^{\ell}.
\end{aligned}
$$

This completes the proof. □

From the above lemma we obtain the following result.

Theorem 1. *For the weight factors given in (29) with $w_1 > w_c$, the probability that the distance between r_1 and r_2 is ℓ as the caterpillar size N goes to infinity is given by*

$$
\psi(\ell) \equiv \lim_{N\to\infty} \frac{Z_{N,\ell}}{Z_N} = (\ell-1)\left(1-\frac{w_1}{w_c}\right)^2 \left(\frac{w_c}{w_1}\right)^{\ell}. \tag{37}
$$

For a given ℓ, exactly one of the vertices on the spine has an infinite order, and the orders of the other vertices are identically and independently distributed by

$$
\phi(k) = \frac{1}{w_c}(k-1)k^{-\beta}, \qquad k \geqslant 2. \tag{38}
$$

Proof. Combining lemma 2 with (33), we obtain (37). If the length of an infinite caterpillar is $\ell < \infty$, it is clear that there is one or more vertices of infinite order. The inequality (36) shows that there can be at most one vertex of infinite order in the limit $N \to \infty$. Finally, the distribution of the orders of the vertices which have a finite order in the thermodynamic limit is obtained by an argument similar to the one leading to equation (18), cf equation (A.7). □

In the appendix, we prove the existence of a measure $\tilde{\nu}$ on the set of infinite caterpillars which describes the subcritical phase and is obtained as the limit of the finite volume measures. The above theorem then implies that the Hausdorff dimension d_H of a random caterpillar in the subcritical phase is almost surely (a.s.) infinite since with probability 1 there is a ball of finite radius which contains infinitely many vertices. Similarly, the spectral dimension is a.s. infinite because a random walk which hits the infinite order vertex returns to the root with probability 0. From the analysis below, one can easily check that the return probability on a randomly chosen subcritical caterpillar τ, $p_\tau(t)$ decays faster than any power of t.

In the remainder of this section, we show how the definition of the spectral dimension in terms of the ensemble average with respect to $\tilde{\nu}$, see (25), leads to a spectral dimension

$$d_s = 2(\beta - 1) \tag{39}$$

in the subcritical phase. We will refer to the unique vertex of infinite order as the 'trap'. If the walk hits the trap, it returns to the root with probability 0. Therefore, the part of the caterpillar beyond the trap is irrelevant for the random walk. When finding the spectral dimension, it is therefore natural to consider the probability that the trap is at a distance ℓ from the root instead of considering the probability of the total length of the caterpillar given in (37).

For a caterpillar of a given length, all the vertices between r_1 and r_2 are equally likely to be of infinite order so the probability that the trap is at a distance ℓ from root is given by

$$p(\ell) = \sum_{k=\ell+1}^{\infty} \frac{\psi(k)}{k-1} = \left(1 - \frac{w_c}{w_1}\right)\left(\frac{w_c}{w_1}\right)^{\ell-1}. \tag{40}$$

From now on, we will disregard the part of the caterpillar beyond the trap. Let $B_{\ell,k}$ be the set of caterpillars with distance ℓ between root and trap and which have one vertex of order k and all other vertices of order no greater than k, with the exception of the trap of course. Let $a(k)$ be the probability that a given vertex on the spine between the root and the trap has order no greater than k. Then

$$a(k) = \sum_{q=2}^{k} \phi(q). \tag{41}$$

The probability that at least one of these vertices has order k and all the others have order no greater than k is then

$$c(k,\ell) = a(k)^{\ell-1} - a(k-1)^{\ell-1}. \tag{42}$$

The average return generating function for the subcritical caterpillars is then

$$Q(x) = \sum_{\ell=1}^{\infty} p(\ell) \sum_{k=2}^{\infty} c(k,\ell) \sum_{\tau \in B_{\ell,k}} \tilde{\nu}(\{\tau | \tau \in B_{\ell,k}\}) Q_\tau(x). \tag{43}$$

For a given distance ℓ between root and trap we denote by M_ℓ the linear subgraph which starts at the root and ends at the trap, see figure 4.

The first return generating function for M_ℓ is given by

$$P_{M_\ell}(x) = 1 - \sqrt{x}\frac{(1+\sqrt{x})^\ell + (1-\sqrt{x})^\ell}{(1+\sqrt{x})^\ell - (1-\sqrt{x})^\ell}, \tag{44}$$

see e.g. [12]. Now attach k links to each vertex of the graph M_ℓ except the root and the trap and denote the resulting graph by $M_{\ell,k}$, see figure 5. Using the methods of [19], we find that the first return generating function for $M_{\ell,k}$ is

$$P_{M_{\ell,k}}(x) = \left(1 + \frac{k}{2}x\right) P_{M_\ell}(x_k(x)), \tag{45}$$

where

$$x_k(x) = \frac{\frac{k^2}{4}x^2 + (1+k)x}{\left(1+\frac{k}{2}x\right)^2}. \tag{46}$$

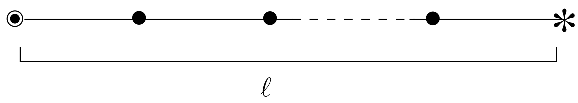

Figure 4. The graph M_ℓ. The root is denoted by a circled vertex and the trap by an asterisk.

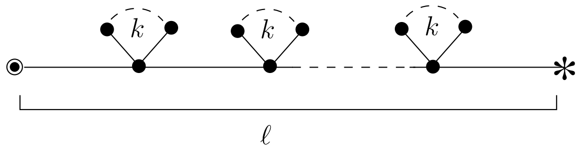

Figure 5. The graph $M_{\ell,k}$.

To find an upper bound on the spectral dimension of subcritical caterpillars, we establish a lower bound on the nth derivative of the average return generating function. Let n be the smallest positive integer such that $Q^{(n)}(x)$ diverges as $x \to 0$. We see in the following calculations that we have to choose n such that $n+1 < \beta \leqslant n+2$. By (23), we find that $(-1)^n Q_\tau^{(n)} \geqslant (-1)^n P_\tau^{(n)}$ for any τ. Thus, by differentiating (43) n times and throwing away every term in the sum over ℓ except $\ell = 2$, we get the lower bound

$$(-1)^n Q^{(n)}(x) \geqslant (-1)^n \left(1 - \frac{w_c}{w_1}\right) \frac{w_c}{w_1} \sum_{k=2}^{\infty} \phi(k) P_{M_{2,k-2}}^{(n)}(x). \tag{47}$$

We easily find that

$$P_{M_{2,k-2}}(x) = \frac{1-x}{2+(k-2)x} \tag{48}$$

and show by induction that

$$P_{M_{2,k-2}}^{(n)}(x) = (-1)^n n! \frac{(k-2)^{n-1}k}{(2+(k-2)x)^{n+1}}. \tag{49}$$

Then, by (38) and (49),

$$\begin{aligned} (-1)^n \sum_{k=2}^{\infty} \phi(k) P_{M_{2,k-2}}^{(n)}(x) &= \frac{n!}{w_c} \sum_{k=2}^{\infty} \frac{(k-2)^{n-1}k^{1-\beta}(k-1)}{(2+(k-2)x)^{n+1}} \\ &\geqslant C x^{\beta-n-2} \int_x^{\infty} \frac{y^{n+1-\beta}}{(2+y)^{n+1}}\, \mathrm{d}y, \end{aligned} \tag{50}$$

where $C > 0$ is a constant. If $\beta < n+2$, the last integral is convergent when $x \to 0$ but if $\beta = n+2$ it diverges logarithmically. In both cases, we get an upper bound for the spectral dimension $d_s \leqslant 2(\beta - 1)$.

To find a lower bound on the spectral dimension of subcritical caterpillars, we establish an upper bound on the nth derivative of the average return generating function. First note that $1 > a(k) = a(k-1) + \phi(k)$ and therefore

$$\begin{aligned} c(k,\ell) &= (a(k) - a(k-1)) \\ &\quad \times (a(k)^{\ell-2} + a(k)^{\ell-3}a(k-1) + \cdots + a(k)a(k-1)^{\ell-3} + a(k-1)^{\ell-2}) \\ &\leqslant \phi(k)(\ell-1). \end{aligned} \tag{51}$$

Now consider a caterpillar $\tau \in B_{\ell,k}$ and the graph M_ℓ. Denote the vertices on the spine of M_ℓ between the root and the trap by $s_1, s_2, \ldots, s_{\ell-1}$. One can obtain the graph τ from M_ℓ by attaching $m_\tau(s_i)$ links to s_i, $i = 1, \ldots, \ell - 1$, where $0 \leqslant m_\tau(s_i) \leqslant k - 2$. Using the methods of [19], we can write

$$Q_\tau(x) = \sum_{\substack{\omega: r_1 \to r_1 \\ \text{on } M_\ell}} K_\tau(x, \omega) W_{M_\ell}(\omega)(1 - x)^{|\omega|/2}, \tag{52}$$

where the sum is over all random walks ω on M_ℓ which begin and end at the root

$$K_\tau(x, \omega) = \prod_{\substack{t=1 \\ \omega_t \in \{s_1, \ldots, s_{\ell-1}\}}}^{|\omega|-1} \left(1 + \frac{m_\tau(\omega_t)}{2} x\right)^{-1}, \tag{53}$$

$$W_{M_\ell}(\omega) = \prod_{t=0}^{|\omega|-1} (\sigma(\omega_t))^{-1}, \tag{54}$$

where ω_t is the vertex at which ω is located at step t and $|\omega|$ denotes the length of ω. The ith derivative of the function $K_\tau(x, \omega)$ can be estimated as

$$(-1)^i \frac{\mathrm{d}^i}{\mathrm{d}x^i} K(x, \omega) \leqslant H(|\omega|) \frac{(k-2)^i}{(2 + (k-2)x)^i}, \tag{55}$$

where H is a polynomial with positive coefficients. From relation (23) and the explicit formula (44), one can easily see that $(-1)^i Q_{M_\ell}^{(i)}(0)$ is a positive polynomial in ℓ of degree $2i + 1$. Therefore, differentiating (52) n times and using the estimate (55), we get the upper bound

$$(-1)^n Q_\tau^{(n)}(x) \leqslant \sum_{i=0}^{n} S_i(\ell) \frac{(k-2)^i}{(2 + (k-2)x)^i}, \tag{56}$$

where the S_i are positive polynomials in ℓ. Differentiating (43) n times w.r.t. x and using the estimates (51) and (56) we finally obtain

$$(-1)^n Q^{(n)}(x) \leqslant \sum_{i=0}^{n} \sum_{\ell=1}^{\infty} p(\ell) S_i(\ell)(\ell - 1) \sum_{k=2}^{\infty} \phi(k) \frac{(k-2)^i}{(2 + (k-2)x)^i}. \tag{57}$$

The sum over ℓ is convergent since S_i is a polynomial in ℓ and $p(\ell)$ decays exponentially. The sum over k is estimated from above by an integral as in (50) which yields a lower bound on the spectral dimension $d_s \geqslant 2(\beta - 1)$. This proves (39).

5. Discussion

In this paper, we have given a description of the phases of the random caterpillar model. However, it is not complete. First of all, in the subcritical nongeneric phase, when $w_1 > w_c$, we limit ourselves to the particular choice of weights in (29). This strict power law can easily be relaxed to an asymptotic power law. It is however not clear how to generalize this to arbitrary weights satisfying $w_1 > w_c$.

Secondly, we have no rigorous results on what happens on the critical line of the phase diagram in figure 3 when $w_1 = w_c$. This problem is discussed in similar models in [6, 8] where it is argued that when $g''(1) < \infty$ the phase is characterized as the generic phase and when $g''(1) = \infty$, the critical exponent of Z_N changes continuously with β.

The order of the phase transition from the condensed phase to the fluid phase also depends on whether $g''(1)$ is finite or infinite. Define the free energy as

$$F(w_1) = \lim_{N \to \infty} \frac{\log Z_N(w_1)}{N}. \tag{58}$$

Using (14), (16) and (30), one finds that

$$F'(w_1) = \begin{cases} \left(\dfrac{1}{\zeta_0^2 g''(w_1\zeta_0(w_1))} + w_1\right)^{-1} & \text{if} \quad w_1 < w_c \\ w_1^{-1} & \text{if} \quad w_1 > w_c \end{cases} \tag{59}$$

and thus

$$\lim_{w_1 \to w_c^-} F'(w_1) = \frac{1}{\frac{w_c^2}{g''(1)} + w_c}. \tag{60}$$

This shows that when $g''(1) < \infty$ the phase transition is first order but when $g''(1) = \infty$ it is continuous in agreement with [6, 8].

The caterpillar model can be generalized to more complicated tree models by replacing the leaves on the spine by trees with vertices of order bounded by K, the caterpillars corresponding to $K = 1$. With similar analysis as for the caterpillars, one obtains two phases: a fluid phase (generic) and a condensed phase (nongeneric), separated by a critical value of w_1 given by

$$w_c(K) = g'(1) - \sum_{n=2}^{K} w_n. \tag{61}$$

In the fluid phase, the finite volume probability measures converge to a measure concentrated on trees with an infinite spine with critical Galton–Watson outgrowths analogous to the generic trees in [13]. In the crumpled phase, the measures converge to trees with spine of a finite length ℓ distributed by

$$\psi(\ell, K) = (\ell - 1)\left(1 - \frac{w_1}{w_c(K)}\right)^2 \left(\frac{w_c(K)}{w_1}\right)^{\ell}. \tag{62}$$

Exactly one of the vertices on the spine has infinite degree and the order of other vertices is independently distributed by

$$\phi(k, K) = \frac{1}{w_c(K)}(k-1)w_k, \qquad k \geqslant 2. \tag{63}$$

The outgrowths from the spine are independent subcritical Galton–Watson trees with offspring probabilities

$$p_n(K) = \frac{w_{n+1}}{\sum_{n=1}^{K} w_n}, \qquad 0 \leqslant n \leqslant K - 1. \tag{64}$$

As $N \to \infty$, one finds that the size of the large vertex is approximately $(1 - m(K))N$, where $m(K) < 1$ is the mean offspring probability of the Galton–Watson process. This is in agreement with analogous results in [6, 8, 20]. What makes the calculations easy in the condensed phase in the above models is the fact that the large vertex which emerges as $N \to \infty$ has to stay on the spine due to the restriction on the order of the vertices in the outgrowths. When the cutoff on the vertex orders is removed ($K = \infty$), one obtains nongeneric trees. In this case, it is more difficult to locate the large vertex and one has to use other methods in the calculations. However, we expect the above characterization of the condensed phase to hold with minor adjustments as is argued in [20]. This will be addressed in a forthcoming paper on nongeneric trees.

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Appendix A. The Gibbs measure in the condensed phase

In this appendix, we consider the set $\tilde{B}$ of all caterpillars defined in (5). We equip this set with a metric and adopt the methods of [11] (see also [2, 7]) to prove the existence of a probability measure on this set which describes the subcritical phase.

We define a metric d on $\tilde{B}$ by

$$d(b,c) = \begin{cases} \max\left\{ \dfrac{1}{1+\min\{b_i,c_i\}} \middle| b_i \neq c_i \right\} & \text{if} \quad \ell(b) = \ell(c), \\ 1 & \text{otherwise} \end{cases} \tag{A.1}$$

where $b = (b_1, b_2, \ldots)$ and $c = (c_1, c_2, \ldots)$. We define the maximum of the empty set to be 0. If $\ell(b) = \ell(c) = \infty$, note that the maximum of $\left\{ \frac{1}{1+\min\{b_i,c_i\}} \middle| b_i \neq c_i \right\}$ is attained since the only possible accumulation point of this set is 0. If $b_i = c_i = \infty$ for some i then $\frac{1}{1+\min\{b_i,c_i\}} = 0$. It is an elementary calculation to verify that this definition fulfils the axioms for a metric.

Denote the open ball centred at b and with radius s by $\mathcal{B}_s(b)$. It is easy to verify that these balls are both open and closed and that if $c \in \mathcal{B}_s(b)$ then $\mathcal{B}_s(c) = \mathcal{B}_s(b)$. Denote the set of caterpillars of fixed length ℓ by $\tilde{B}^{(\ell)}$. For any $\ell \in \mathbb{N}$, the set $\tilde{B}^{(\ell)}$ is compact. Define

$$\tilde{B}' = \bigcup_{N=1}^{\infty} \tilde{B}_N. \tag{A.2}$$

The set $\tilde{B}'$ is a countable dense subset of $\tilde{B}$.

From now on we consider the weight factors (29) with $w_1 > w_c$. The probability measures $\tilde{\nu}_N$ on $\tilde{B}_N$ will be shown to converge to a measure $\tilde{\nu}$ on $\tilde{B}$.

Theorem A1. *For the weight factors (29) with $w_1 > w_c$, the measures $\tilde{\nu}_N$ viewed as probability measures on $\tilde{B}$ converge weakly to a measure $\tilde{\nu}$ as $N \to \infty$ and $\tilde{\nu}$ is concentrated on the set of caterpillars of finite length with exactly one vertex of infinite order. The length of the spine is distributed by (37). All the vertices between r_1 and r_2 are equally likely to be of infinite order and the orders of the others are independently distributed by (38).*

Proof. Applying the methods of [11], we need to show the following.

(i) The sequence $\left(\tilde{\nu}_N\left(\mathcal{B}_{\frac{1}{k}}(b)\right)\right)_{N=1}^{\infty}$ converges for all $k \in \mathbb{N}$ and all $b \in \tilde{B}'$.

(ii) For every $\epsilon > 0$, there exists a compact subset $C \subseteq \tilde{B}$ such that

$$\tilde{\nu}_N(\tilde{B} \setminus C) < \epsilon, \qquad \text{for all} \quad N \in \mathbb{N}. \tag{A.3}$$

To prove property (i), take a finite caterpillar $b = (b_1, \ldots, b_{\ell(b)-1}) \in \tilde{B}'$. In order to streamline the notation, we write $\ell(b) = \ell$. Denote the set of indices i for which $b_i < k$ by $\underline{I}$ and the set of indices i for which $b_i \geqslant k$ by $\overline{I}$. Then,

$$\mathcal{B}_{\frac{1}{k}}(b) = \{c \in \tilde{B}^{(\ell)} | c_i = b_i \text{ if } i \in \underline{I}, c_i \geqslant k \text{ if } i \in \overline{I}\}. \tag{A.4}$$

Denote the number of elements in $\overline{I}$ by R. Now order the indices in $\overline{I}$ in increasing order and for a given caterpillar in $\mathcal{B}_{\frac{1}{k}}(b)$ let N_i, $1 \leqslant i \leqslant R$ be the term in the caterpillar corresponding to the ith index in $\overline{I}$. We can then write

$$\tilde{\nu}_N\left(\mathcal{B}_{\frac{1}{k}}(b)\right) = Z_N^{-1} w_1^{N-\ell} W_0 \sum_{\substack{N_1+\cdots+N_R=N+\ell-2-b_0 \\ N_m \geqslant k, \forall m}} \prod_{i=1}^{R} \left[\left(N_i - 1\right) w_{N_i}\right], \tag{A.5}$$

where

$$b_0 = \sum_{i \in \underline{I}} b_i \qquad \text{and} \qquad W_0 = \prod_{i \in \underline{I}} \left[(b_i - 1) w_{b_i} \right].$$

First note that if $\overline{I}$ is empty then $\tilde{\nu}_N\left(\mathcal{B}_{\frac{1}{k}}(b)\right) \longrightarrow 0$ when $N \longrightarrow \infty$. If it is not empty, there exists an index $i \in \overline{I}$ in the above sum such that $N_i \geqslant \frac{N+\ell-2-b_0}{R}$. If there is another index $j \neq i$ such that $N_j > C$ where $C \geqslant k$ is a constant then we get an upper bound

$$K \sum_{N_2 > C} (N_2 - 1)\, w_{N_2} \tag{A.6}$$

on that contribution to the above sum using (30) and the methods in the proof of lemma 2 where K is a positive number which only depends on b and k. The last expression goes to zero as $C \longrightarrow \infty$ since $g'(1)$ is finite. Estimating the remaining contribution to (A.5), we get

$$\begin{aligned}
(w_1 - w_c)^2 w_1^{-\ell} N^{\beta-1} W_0 \sum_{j=1}^{R} \sum_{\substack{N_1+\cdots+N_R = N+\ell-2-b_0 \\ k \leqslant N_m \leqslant C, \quad m \neq j}} \prod_{i \in \overline{I}} \left[\left(N_i - 1 \right) w_{N_i} \right) \right] (1 + o(1)) \\
\underset{N\to\infty}{\longrightarrow} (w_1 - w_c)^2 w_1^{-\ell} W_0 R \left(\sum_{n=k}^{C} (n-1) w_n \right)^{R-1} \\
\underset{C\to\infty}{\longrightarrow} (w_1 - w_c)^2 w_1^{-\ell} W_0 R \left(\sum_{n=k}^{\infty} (n-1) w_n \right)^{R-1}
\end{aligned} \tag{A.7}$$

proving the convergence. The calculations show that the measure is concentrated on the set of caterpillars with exactly one infinite term.

In order to prove property (ii), we take our compact set to be

$$C_L = \bigcup_{\ell=1}^{L} \tilde{B}^{(\ell)} \tag{A.8}$$

and we need to show that

$$\tilde{\nu}_N(\{b \in \tilde{B} \mid \ell(b) > L\}) \longrightarrow 0 \qquad \text{as} \qquad L \longrightarrow \infty \tag{A.9}$$

uniformly in N. We estimate as in the proof of lemma 2

$$\tilde{\nu}_N(\{b \in \tilde{B} \mid \ell(b) = \ell\}) = \frac{Z_{N,\ell}}{Z_N} \leqslant C \left(\frac{w_c}{w_1} \right)^{\ell} (\ell - 1)^{\beta},$$

where C is a constant. Since $w_1 > w_c$ this completes the proof of the convergence. The distribution of the length of the spine and order of vertices follows from (A.7). □

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